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# Weibull master curves and fracture toughness testing

## Part I *Master curves for quasi-static uniaxial tensile and bend tests*

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Three types of specimen-size-independent Weibull master curves, characterizing strength and failure of macroscopically homogeneous, brittle materials have been derived. These Weibull master curves are significant if an uniaxial tensile stress is applied to the investigated specimens, as, for example, in the case of quasi-static uniaxial tensile tests or, under some restrictions, in the case of quasi-static three- or four-point bend tests. In addition, the existence of three types of apparent fracture toughness master curves, which can be applied to any material undergoing brittle cleavage fracture such as ceramics, intermetallics, or structural steels at low homologous temperatures, has been established. Furthermore, the same is also valid for the specimen-size-independent Weibull master curves. The apparent fracture toughness master curves can be obtained, by performing fracture toughness tests, or simply by applying a mathematical transformation to the corresponding Weibull master curves, which have been evaluated from quasi-static uniaxial tensile or bend tests. 

**List of Symbols**

| | |
|---|---|
| $c$ | Constant |
| $\sigma$ | Applied failure stress |
| $P(\sigma)$ | Three-parameter, cumulative Weibull failure probability distribution function |
| $\sigma_0$ | Normalizing factor in dimensions of stress |
| $\sigma_\tau$ | Threshold stress, below which no failure occurs |
| $\sigma_{in}$ | Failure stress at the inflexion point $P(\sigma)$ |
| $\bar{\sigma}$ | Mean failure stress |
| $m$ | Weibull modulus |
| $z$ | Distinct value of the cumulative failure probability distribution function |
| $\sigma_z$ | Failure stress corresponding to the cumulative failure probability $z$ |
| $K_I$ | Failure stress intensity |
| $P(K_I)$ | Cumulative failure probability distribution function in terms of $K_I$ |
| $K_{\min}$ | Threshold stress intensity, below which no failure occurs |
| $K_{Iin}$ | Failure stress intensity at the inflexion point of $P(K_I)$ |
| $\bar{K}_I$ | Mean failure stress intensity |
| $K_{Iz}$ | Failure stress intensity corresponding to the cumulative failure probability $z$ |

$I(x, m)$, $K(y, m)$ and $M[e(z), m]$:
Three different types of Weibull master curves

$I\exp(x, m)$, $K\exp(y, m)$ and $M\exp[e(z), m]$:
Three different types of experimental Weibull master curves

$N(a)$, $T(b)$ and $L[d(z)]$:
Three different types of master curves for brittle cleavage fracture toughness testing, i.e., three different types of theoretical, apparent fracture toughness master curves

$N\exp(a)$, $T\exp(b)$ and $L\exp[d(z)]$:
Three different types of experimental, apparent fracture toughness master curves

$x$, $y$ and $e(z)$:
Different types of scaled failure stresses

$a$, $b$ and $d(z)$:
Different types of scaled failure stress intensities

$x_{cr}$, $y_{cr}$, $e_{cr}(z)$, $a_{cr}$, $b_{cr}$ and $d_{cr}(z)$:
Values of the cross-over points formed by the corresponding experimental and theoretical master curves

## 1. Introduction

In an earlier paper, Lambrigger [1] derived specimen-size-independent Weibull master curves $I(x, m)$, existing for Weibull moduli $m > 1$, and alternative Weibull master curves $K(y, m)$, existing for every real $m > 0$. $I(x, m)$ and $K(y, m)$ represent scaled cumulative failure probability distribution functions; the former type $I(x, m)$ was obtained by scaling the three-parameter cumulative Weibull failure probability distribution function $P(\sigma)$, with the stress-value of the corresponding inflexion point $\sigma_{in}$, the latter type $K(y, m)$ by scaling $P(\sigma)$ with the mean stress $\bar{\sigma} = \int_0^1 \sigma dP$. According to Weibull [2, 3], $P(\sigma)$ is always significant if an uniaxial tensile stress is applied to materials undergoing brittle cleavage fracture, as, for example, in quasi-static uniaxial tensile tests. Such brittle materials might be ceramics, semiconductors, intermetallics, or even structural steels in the brittle low-temperature range. The following transformations have been proved to be valid [1, 4]:

$$P(\sigma) = 1 - \exp\left\{-\left(\frac{\sigma - \sigma_\tau}{\sigma_0}\right)^m\right\}; \quad \sigma > \sigma_\tau \qquad (1)$$

$$P(\sigma) = I(x, m) = 1 - \exp\left[\frac{(1-m)}{m} x^m\right] \qquad (2)$$

$$x = \frac{\sigma - \sigma_\tau}{\sigma_{in} - \sigma_\tau} \qquad (3)$$

$$\sigma_{in} = \sigma_0\left[\frac{m-1}{m}\right]^{1/m} + \sigma_\tau \qquad (4)$$

whereby $\sigma$ denotes the applied failure stress, $P(\sigma)$ the three-parameter cumulative Weibull failure probability distribution function, $\sigma_\tau$ the threshold stress underneath $P(\sigma)$ is equal to zero, $\sigma_0$ a normalizing factor that has dimensions of stress, $m$ the Weibull modulus, $x$ the scaled stress, and $\sigma_{in}$ the stress-value of the inflexion point of $P(\sigma)$. The alternative scaling parameter $\bar{\sigma}$ for the master curves $K(y, m)$ can be calculated as follows:

$$\begin{aligned}
\bar{\sigma} &= \int_0^1 \sigma dP = \sigma_\tau + \int_0^1 (\sigma - \sigma_\tau)\, dP \\
&= \sigma_\tau + m \int_0^\infty \exp\left\{-\left[\left(\frac{\sigma - \sigma_\tau}{\sigma_0}\right)^m\right]\right\} \\
&\quad \times \left(\frac{\sigma - \sigma_\tau}{\sigma_0}\right)^m d\sigma \\
&= \sigma_\tau + \sigma_0 \int_0^\infty \left[\left(\frac{\sigma - \sigma_\tau}{\sigma_0}\right)^m\right]^{1/m} \\
&\quad \times \exp\left\{-\left[\left(\frac{\sigma - \sigma_\tau}{\sigma_0}\right)^m\right]\right\} d\left[\left(\frac{\sigma - \sigma_\tau}{\sigma_0}\right)^m\right] \\
&= \sigma_\tau + \sigma_0 \Gamma\left(1 + \frac{1}{m}\right) \qquad (5)
\end{aligned}$$

whereby the complete Gamma-function $\Gamma[1 + (1/m)]$ is defined by

$$\Gamma\left(1 + \frac{1}{m}\right) = \int_0^\infty u^{1/m} \exp(-u)\, du \qquad (6)$$

Because the complete Gamma-function $\Gamma[1+(1/m)]$ is real and positive for every $m > 0$, it is easily understood from Equation 5 that $\bar{\sigma}$ is also real and positive in all these cases. Trying to achieve the alternative variable transformation

$$y = \frac{\sigma - \sigma_\tau}{\bar{\sigma} - \sigma_\tau} \qquad (7)$$

the alternative Weibull master curves $K(y, m)$ can be obtained for every $m > 0$ by isolating $\sigma - \sigma_\tau$ in Equation 7, replacing $\bar{\sigma}$ with the help of Equation 5, and by substituting the resulting expression for $\sigma - \sigma_\tau$ in Equation 1:

$$P(\sigma) = K(y, m) = 1 - \exp\left\{-\left[\Gamma\left(1 + \frac{1}{m}\right)\right]^m y^m\right\} \qquad (8)$$

Weibull master curves $I(x, m)$ are also applicable to formal cases, where the Weibull modulus $m$ is smaller than zero, because $\sigma_{in}$ is, following Equation 4, also real and positive for $m < 0$. Lambrigger [5] has shown that modified Weibull master curves representing scaled, cumulative critical crack size distribution functions, can exhibit negative $m$-values. The master curves $K(y, m)$, however, are less useful in the formal cases, where $m$ is negative. Furthermore, $\Gamma[1 + (1/m)]$ is negative for an infinite number of extended ranges in the interval $-1 < m < 0$, whereby it remains always positive for $m < -1$. The master curves $I(x, m)$ and $K(y, m)$ are displayed in Figs 1 and 2.

If $m > 1$, $I(x, m)$ can be calculated from $K(y, m)$ by setting $x = y = c =$ constant and vice versa:

$$I(c, m) = 1 - \exp\left\{\frac{(m-1)\, ln[1 - K(c, m)]}{m\left(\Gamma\left(1 + \frac{1}{m}\right)\right)^m}\right\} \qquad (9)$$

Equation 9, displayed in Fig. 3, expresses the possibility of comparing two corresponding Weibull master curves $I(x, m)$ and $K(y, m)$ at any positive, fixed value of the variables $x$ resp. $y$. This is achieved, formally, by setting $x = y = c =$ constant. Inspection of Equation 9 provides the following results: $I(c, m_o) = K(c, m_o)$ for $m_o \approx 3.31$, $I(c, m) > K(c, m)$ for Weibull moduli $m > m_o$, and $I(c, m) < K(c, m)$ for $1 < m < m_o$.

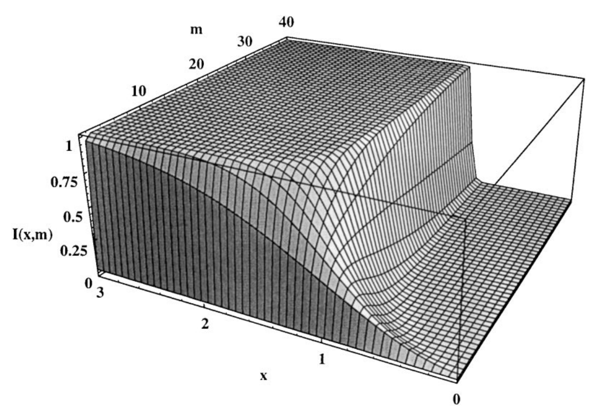


*Figure 1* Weibull master curves $I(x, m)$ as functions of the scaled failure stress $x$ and the Weibull modulus $m$.

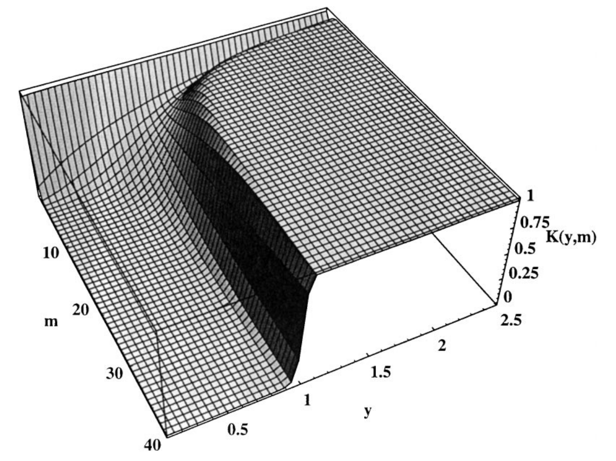


*Figure 2* Alternative Weibull master curves $K(y, m)$ as functions of the scaled failure stress $y$ and the Weibull modulus $m$.

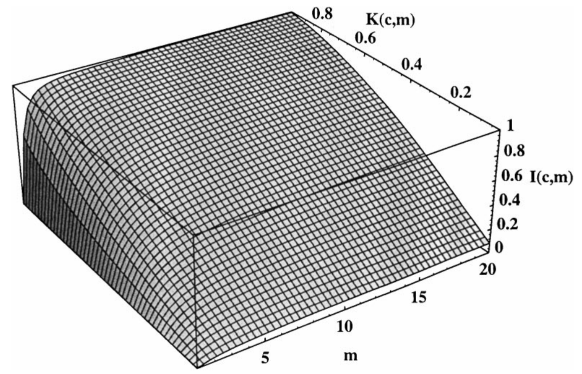


*Figure 3* Weibull master curves $I(c, m)$ as functions of the alternative Weibull master curves $K(c, m)$ and the Weibull modulus $m$.

The means of the transformed variables $x$ and $y$ ($\bar{x}$ and $\bar{y}$), as well as the values of the inflexion points of $I(x, m)$ and $K(y, m)$ ($x_{in}$ resp. $y_{in}$), are given by

$$\bar{x} = \int_0^1 x\, dI = \frac{\Gamma\left(1 + \frac{1}{m}\right)}{\left(\frac{m-1}{m}\right)^{1/m}} \tag{10}$$

$$\bar{y} = \int_0^1 y\, dK = 1 \tag{11}$$

$$x_{in} = 1 \tag{12}$$

$$y_{in} = \frac{\left(\frac{m-1}{m}\right)^{1/m}}{\Gamma\left(1 + \frac{1}{m}\right)} = \frac{1}{\bar{x}} \tag{13}$$

Furthermore, the following equalities hold:

$$\begin{aligned} I(\bar{x}, m) &= K(\bar{y}, m) = P(\bar{\sigma}, m) \\ &= 1 - \exp\left\{-\left[\left(\Gamma\left(1 + \frac{1}{m}\right)\right)^m\right]\right\} \end{aligned} \tag{14}$$

$$\begin{aligned} I(x_{in}, m) &= K(y_{in}, m) = P(\sigma_{in}, m) \\ &= 1 - \exp\left(\frac{1-m}{m}\right) \end{aligned} \tag{15}$$

Weibull [2, 3] has shown that $m$ is a specimen-size-independent magnitude, if an uniaxial stress is applied to macroscopically homogeneous, brittle materials. Therefore, the same is also true for the one-parameter master curves $I(x, m)$ and $K(y, m)$. Moreover, Lambrigger [4] has also shown that experimental Weibull master curves $I\exp(x, m)$ and $K\exp(y, m)$ of materials undergoing an amount of stable crack growth prior to failure, such as 8 wt % yttria partially stabilized zirconia/20 vol % $\beta$-alumina composites (8 wt % Y-PSZ/20 vol % $\beta$-alumina composites), enable the characterization of the toughening mechanisms operating in the investigated materials. It has been found that experimental Weibull master curves $I\exp(x, m = 7)$ and $K\exp(y, m = 7)$ can be constructed for a 8 wt % Y-PSZ/20 vol % $\beta$-alumina composite by calculating the Weibull modulus $m$ from the upper $\sigma$-range of experimental cumulative failure stress probability distributions $P(\sigma_i)$.

## 2. Apparent fracture toughness master curves

It has already been recognized by Weibull [2, 3] that the critical event for brittle cleavage fracture is the propagation of a microcrack starting from a critical defect, and that this event can be described by a statistical weakest link model. A material-, specimen-size-, and testing-temperature-dependent cumulative failure probability distribution function, which is most appropriate to display brittle failure data of fracture toughness testing, has been discussed extensively by Wallin [6–8]. It deals with a Weibull type equation which, in terms of the applied stress intensity $K_I$, is given by

$$P(K_I) = 1 - \exp\{-c[K_I - K_{\min}]^4\} \tag{16}$$

whereby $P(K_I)$ defines the cumulative failure probability distribution function of the investigated specimen, $K_{\min}$ the stress intensity underneath the cumulative failure probability is zero, and $c$ represents a material-, specimen-size-, and testing-temperature-dependent constant. Wallin [6, 7] has shown that Equation 16 can be applied to every type of brittle material under all circumstances. Therefore, parameter-free master curves $N(a)$ and $T(b)$ can be calculated. $N(a)$ is received by scaling $P(K_I)$, with the stress intensity value of its inflexion point $K_{Iin}$. The following reparametrization is thus possible:

$$a = \frac{K_I - K_{\min}}{K_{Iin} - K_{\min}} \tag{17}$$

$$P(K_I) = N(a) = 1 - \exp\left[\left(-\frac{3}{4}\right)a^4\right] \tag{18}$$

The master curve for brittle cleavage fracture toughness testing $N(a)$ is formally equal to $I(x, m = 4)$. Thus, $N(a)$ is material-, specimen-size-, and testing-temperature-independent. By setting $x = a = c =$ constant, the following equation has been shown to be valid [9]:

$$\begin{aligned} N(c) = 1 - \exp\Bigg\{&\left(\frac{-3}{4}\right)\left(\frac{-m}{1-m}\right)^{4/m} \\ &\times \{-ln[1 - I(c, m)]\}^{4/m}\Bigg\} \end{aligned} \tag{19}$$

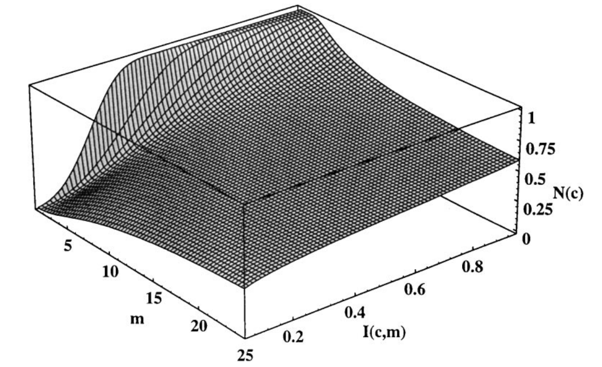


*Figure 4* The master curve for brittle cleavage fracture toughness testing $N(c)$ as a function of the Weibull master curves $I(c, m)$ and the Weibull modulus $m$.

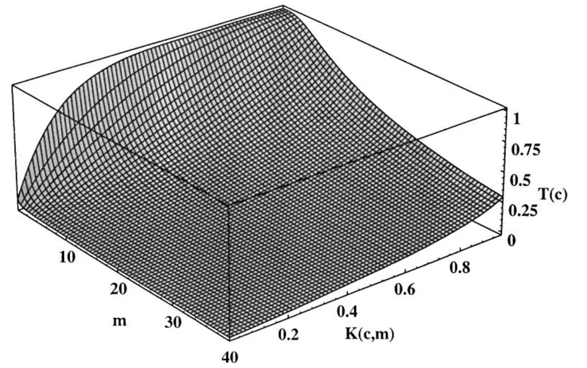


*Figure 5* Alternative master curve for brittle cleavage fracture toughness testing $T(c)$ as a function of the alternative Weibull master curves $K(c, m)$ and the Weibull modulus $m$.

Equation 19, displayed in Fig. 4, enables the calculation of single values $(a_i, N(a_i))$ from transformed, experimental failure data points $(x_i, I(x_i, m))$ of uniaxial tensile tests. The uniaxial tensile tests provide the failure data points $(\sigma_i, P(\sigma_i))$, $\sigma_i$ being the strength (failure stress) of the specimen $i$ and $P(\sigma_i)$ representing the corresponding experimental cumulative failure probability distribution. The experimental, apparent fracture toughness master curve $N\exp(a)$ is then received by fitting a curve to the data points $(a_i, N(a_i))$. Furthermore, the experimental, specimen-size-independent failure data points $(x_i, I(x_i, m))$, the single values $(a_i, N(a_i))$ and $N\exp(a)$ can also be derived from experimental, cumulative failure stress distributions $P(\sigma_i)$ of quasi-static three- or four-point bend tests according to Weibull [2, 3], if the threshold stress $\sigma_\tau$ is equal to zero. Thus, $N\exp(a)$ is determined without having performed proper fracture toughness tests.

The theoretical, alternative apparent fracture toughness master curve $T(b)$, is achieved by eliminating the two Weibull parameters $c$ and $K_{\min}$, as well as scaling with the mean-stress intensity $\bar{K}_I = \int_0^1 K_I \, dP$. The alternative master curve for brittle cleavage fracture toughness testing $T(b)$, is then formally equivalent to $K(y, m = 4)$. However, in this case the variable $y$ is defined by the equation

$$y = \frac{(K_I - K_{\min})}{(\bar{K}_I - K_{\min})} = b \qquad (20)$$

whereby

$$\begin{aligned}
\bar{K}_I &= \int_0^1 K_I \, dP = K_{\min} + \int_0^1 (K_I - K_{\min}) \, dP \\
&= K_{\min} + \int_0^\infty 4c \exp\{-[c(K_I - K_{\min})^4]\} \\
&\quad \times (K_I - K_{\min})^4 \, dK_I \\
&= K_{\min} + \frac{1}{c^{1/4}} \int_0^\infty [c(K_I - K_{\min})^4]^{1/4} \exp \\
&\quad \times \{-[c(K_I - K_{\min})^4]\} \, d[c(K_I - K_{\min})^4] \\
&= K_{\min} + \frac{\Gamma\left(1 + \frac{1}{4}\right)}{c^{1/4}} \qquad (21)
\end{aligned}$$

The complete Gamma-function $\Gamma[1 + (1/4)]$ is given as follows:

$$\Gamma\left(1 + \frac{1}{4}\right) = \int_0^\infty u^{1/4} \exp(-u) \, du \approx 0.906 \qquad (22)$$

The theoretical, alternative apparent fracture toughness master curve $T(b)$ is then defined by

$$\begin{aligned}
P(K_I) &= T(b) = K(y = b, m = 4) \\
&= 1 - \exp\left\{-\left[\Gamma\left(1 + \frac{1}{4}\right)\right]^4 b^4\right\} \qquad (23)
\end{aligned}$$

Moreover, by setting $y = b = c =$ constant resp. $a = b = c =$ constant, the following equations can be derived:

$$\begin{aligned}
T(c) = 1 - \exp&\left\{-\left(\Gamma\left(1 + \frac{1}{4}\right)\right)^4\right. \\
&\left.\times \left\{-\left[\Gamma\left(1 + \frac{1}{m}\right)\right]^{-m} ln[1 - K(c, m)]\right\}^{4/m}\right\}
\end{aligned} \qquad (24)$$

$$T(c) = 1 - \exp\left\{\frac{3}{4\left(\Gamma\left(1 + \frac{1}{4}\right)\right)^4} ln[1 - N(c)]\right\} \qquad (25)$$

Equation 25, displayed in Fig. 5, expresses the possibility of comparing two corresponding apparent fracture toughness master curves $N(a)$ and $T(b)$ at any positive, fixed value of the variables $a$ resp. $b$. This is done, formally, by setting $a = b = c =$ constant. Further, the following is valid: $N(c) > T(c)$ for $0 < c < \infty$. On the other side, Equation 24 enables the calculation of single values $(b_i, T(b_i))$ from transformed, experimental failure data points $(y_i, K(y_i, m))$ of uniaxial tensile tests. Again, the experimental, apparent fracture toughness master curve $T\exp(b)$ can be obtained by fitting a curve to the data points $(b_i, T(b_i))$.

## 3. Evaluation of the Weibull parameters

In order to calculate Weibull master curves, which are independent of $\sigma_\tau$ and $\sigma_0$ and only a function of the shape factor $m$, usually three failure data points

$i$, $i+1$ and $i+2$ have to be pointed out. The cumulative Weibull probability distribution function at the point $i$, is then given by

$$P(\sigma_i) = 1 - \exp\left\{-\left[\frac{\sigma_i - \sigma_\tau}{\sigma_0}\right]^m\right\}; \quad \sigma_i > \sigma_\tau \quad (26)$$

However, if $\sigma_\tau$ has already been evaluated or if $\sigma_\tau \approx 0$, a typical Weibull plot consisting of only two failure data points provides the Weibull modulus $m$ graphically. The Weibull plots are accomplished by plotting log log $[1/\{1-P(\sigma_i)\}]$ as an ordinate and log $(\sigma_i - \sigma_\tau)$ as its abscissa as has been suggested by Weibull [2]. The slope of the straight line obtained by a linear regression analysis of the upper $\sigma$-range represents then the Weibull modulus $m$ [9], as can be seen from Equation (27).

$$\log\log\left[\frac{1}{1-P(\sigma_i)}\right] = m\log[\sigma_i - \sigma_\tau] - m\log\sigma_0 \quad (27)$$

Later, the other parameters can be calculated or checked as follows:

$$\sigma_0 = \frac{(\sigma_{i+1} - \sigma_i)}{\left\{\{-ln[1 - P(\sigma_{i+1})]\}^{1/m} - \{-ln[1 - P(\sigma_i)]\}^{1/m}\right\}} \quad (28)$$

$$\sigma_\tau = \sigma_i - \sigma_0\{-ln[1 - P(\sigma_i)]\}^{1/m} \quad (29)$$

However, if $\sigma_\tau$ is unknown, a difference quotient method developed by Lambrigger [9] and based on an analysis of at least three data points has to be applied, in order to evaluate $m$. Nevertheless, it has been observed that for materials undergoing an amount of stable crack growth prior to failure, normally no clear threshold $\sigma_\tau$ exists. It can be assumed that for such materials no significant lower bound $\sigma_\tau$ can be evaluated, simply because any applied stress promotes at least a change of the microstructure with regard to the initial defects by activation of any toughness mechanism. However, as will be shown in the following sections, this effect is efficiently taken into account by defining deviation parameters for the experimental master curves. Thus, there is no need for the introduction of a threshold stress under these circumstances.

## 4. The general type of master curves

Specimen-size-independent Weibull master curves of a general type, which are obtained by scaling $P(\sigma)$ with any stresses $\sigma_z$ corresponding to a fixed, cumulative failure probability $z$, can formally be constructed for every real $m$. Therefore, the scaling parameter $\sigma_z$ has to be evaluated first in these cases. The following equation is a primary conclusion of the definition:

$$P(\sigma_z) = z \quad (30)$$

By combining Equations 1 and 30, $\sigma_z$ can be obtained through

$$\sigma_z = \sigma_0[-ln(1 - z)]^{1/m} + \sigma_\tau \quad (31)$$

If the variable transformation

$$e(z) = \frac{\sigma - \sigma_\tau}{\sigma_z - \sigma_\tau} \quad (32)$$

has been accomplished, the Weibull master curves $M[e(z), m]$ can be obtained by isolating $\sigma - \sigma_\tau$ in Equation 32, replacing $\sigma_z$ with the help of Equation 31, and by substituting the resulting expression for $\sigma - \sigma_\tau$ in Equation 1:

$$\begin{aligned} P(\sigma) = M[e(z), m] &= 1 - \exp\{ln(1 - z)[e(z)]^m\} \\ &= 1 - (1 - z)^{[e(z)]^m} \end{aligned} \quad (33)$$

In the case of $z = 0.5$, for instance, the Weibull master curves $M[e(z), m]$ are given by

$$M[e(z = 0.5), m] = 1 - (0.5)^{[e(z=0.5)]^m} \quad (34)$$

Equation 34 is displayed in Fig. 6. Experimental Weibull master curves $M\exp[e(z), m]$ are obtained, by fitting a curve to the transformed failure data points $(e_i(z), M[e_i(z), m])$. $M[e_i(z), m]$ and $e_i(z)$ can be calculated from the original failure data points of uniaxial tensile or bend tests $(\sigma_i, P(\sigma_i))$, by using Equations 26, 32, and 33.

The master curves for brittle cleavage fracture toughness testing, which have already been described in a more restricted way [9, 10], are again formally special cases of $M[e(z), m]$, that is $M[e(z), m = 4]$. If the transformed failure data points of uniaxial tensile or bend tests obey specimen-size-independent Weibull master curves $M[e(z), m]$, the results of fracture toughness tests of such brittle materials are given by the master curves for brittle cleavage fracture toughness testing $M[e(z), m = 4]$. In this case, however, the variable $e(z)$ is denoted by $d(z)$ and defined by the equation

$$d(z) = \frac{(K_I - K_{\min})}{(K_{Iz} - K_{\min})} \quad (35)$$

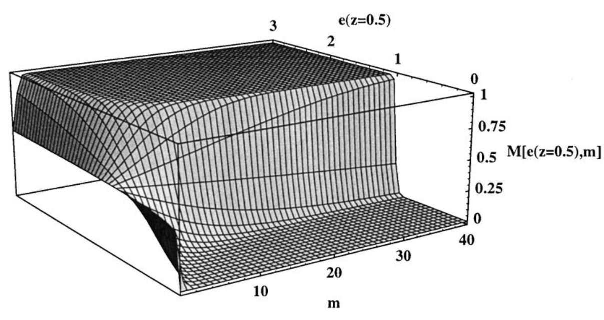


*Figure 6* General type of Weibull master curves $M[e(z = 0.5), m]$, being scaled with the stress value corresponding to a cumulative failure probability $z = 0.5$, as a function of the scaled stress $e(z = 0.5)$ and the Weibull modulus $m$.

whereby $K_I$ denotes the applied stress intensity, $K_{\min}$ the stress intensity underneath the cumulative failure probability $P(K_I)$ is zero, and $K_{Iz}$ the stress intensity value corresponding to the cumulative failure probability $z = P(K_{Iz})$.

From now on, the master curves for brittle cleavage fracture toughness testing $M[e(z) = d(z), m = 4]$ will be denoted by $L[d(z)]$. By setting $e(z) = d(z) = c =$ constant, the following transformation equation is obtained:

$$L[c] \;=\; 1 - \exp\big\{[-ln(1-z)]^{\frac{(m-4)}{m}} \times [-ln(1 - M[c,m])]^{4/m}\big\} \tag{36}$$

Again, Equation 36 enables the calculation of single values $(d_i(z), L[d_i(z)]$, from transformed experimental failure data points $(e_i(z), M[e_i(z), m])$ of quasi-static uniaxial tensile or bend tests. $L\exp[d(z)]$ is finally obtained by fitting a curve to $(d_i(z), L[d_i(z)])$. In addition, the following main result can be obtained by comparing the Weibull master curves $I(x, m)$ to $M[e(z), m]$ for $m > 1$ or $m < 0$ (for $0 < m < 1$, $P(\sigma)$ exhibits no inflexion point for real stresses) by setting $x = e(z) = c =$ constant:

$$M[c,m] \;=\; 1 - \exp\left\{\frac{m}{(1-m)} ln(1-z) \times ln[1 - I(c,m)]\right\} \tag{37}$$

Inspection of equation (37) provides for $e(z) = 0.5$: $I(x, m_1) = M[e(z) = 0.5, m_1]$ for $m_1 \approx 3.26$, $I(x, m) > M[e(z) = 0.5, m]$ for $m > m_1$ and $I(x, m) < M[e(z) = 0.5, m]$ for $1 < m < m_1$.

The means representing the first moments of the variables $e(z)$, denoted by $\overline{e}(z)$, as well as the inflexion points of $M[e(z), m]$, denoted by $e_{in}(z)$, are thoroughly characterizing the Weibull master curves $M[e(z), m]$. They are calculated in the following way:

$$\frac{\partial^2 M[e_{in}(z), m]}{\partial [e(z)]^2} = 0 \tag{38}$$

$$e_{in}(z) = \left\{\frac{m-1}{m[-ln(1-z)]}\right\}^{1/m} \tag{39}$$

whereby $e_{in}(z)$ is real and positive for $m > 1$ or $m < 0$. The means $\overline{e}(z)$ exist for every $m > 0$ or $m < -1$ and are given by

$$\begin{aligned}\overline{e}(z) &= \int_0^1 e(z)\, dM[e(z), m] \\ &= -m[ln(1-z)] \int_0^\infty [e(z)]^m (1-z)^{[e(z)]^m}\, d[e(z)] \\ &= \frac{\Gamma\left(1 + \frac{1}{m}\right)}{[-ln(1-z)]^{1/m}}\end{aligned} \tag{40}$$

Furthermore, the following equalities hold:

$$\begin{aligned}M[e_{in}(z), m] &= I(x_{in}, m) = K(y_{in}, m) \\ &= 1 - \left[(1-z)^{\left\{\frac{m-1}{m[-\ln(1-z)]}\right\}}\right] \\ &= 1 - \exp\left(\frac{1-m}{m}\right)\end{aligned} \tag{41}$$

$$\begin{aligned}M[\overline{e}(z), m] &= I(\overline{x}, m) = K(\overline{y}, m) \\ &= 1 - (1-z)^{[\overline{e}(z)]^m} \\ &= 1 - \exp\{ln(1-z)[\overline{e}(z)]^m\} \\ &= 1 - \exp\left\{-\left[\left(\Gamma\left(1 + \frac{1}{m}\right)\right)^m\right]\right\}\end{aligned} \tag{42}$$

Moreover, we find for the specimen-size-independent Weibull master curves $K(y, m)$ by setting $y = e(z) = x = c =$ constant:

$$M[c,m] = 1 - \exp\left\{\frac{[-ln(1-z)]\, ln[1 - K(c,m)]}{\left[\Gamma\left(1 + \frac{1}{m}\right)\right]^m}\right\} \tag{43}$$

Equation 43 can always be used for $m > 0$ or $m < -1$ because the values of $M[c, m]$ and $K(c, m)$ are significant and lie between zero and one for both master curves in both cases. Inspection of Equation (43) provides the following results for $e(z) = 0.5$: $K(y, m_2) = M[e(z) = 0.5, m_2]$ for $m_2 \approx 3.44$, $K(y, m) < M[e(z) = 0.5, m]$ for $m > m_2$ and $K(y, m) > M[e(z) = 0.5, m]$ for $0 < m < m_2$.

The means of the transformed variables $d(z)$, as well as the $d(z)$-values of the inflexion points of the master curves $L[d(z)]$, denoted by $\bar{d}(z)$ resp. $d_{in}(z)$, characterize the master curves $L[d(z)]$ thoroughly. They are calculated as follows:

$$\frac{\partial^2 L[d_{in}(z)]}{\partial [d(z)]^2} = 0 \tag{44}$$

$$d_{in}(z) = \left[\frac{3}{4[-ln(1-z)]}\right]^{1/4} \tag{45}$$

$$\begin{aligned}\bar{d}(z) &= \int_0^1 d(z)\, dL[d(z)] \\ &= -4[ln(1-z)] \int_0^\infty [d(z)]^4 (1-z)^{[d(z)]^4}\, d[d(z)] \\ &= \frac{\Gamma\left(1 + \frac{1}{4}\right)}{[-ln(1-z)]^{1/4}}\end{aligned} \tag{46}$$

Furthermore, the following equalities hold:

$$\begin{aligned}L[d_{in}(z)] &= 1 - \left[(1-z)^{\left\{\frac{3}{4[-ln(1-z)]}\right\}}\right] \\ &= 1 - \exp\left(-\frac{3}{4}\right) \approx 0.528\end{aligned} \tag{47}$$

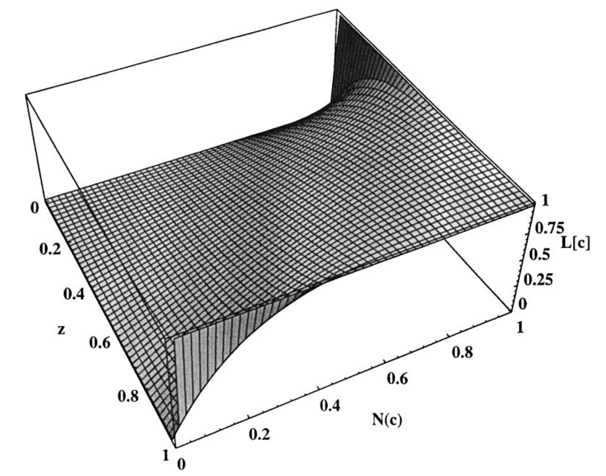


*Figure 7* The general type of master curve for brittle cleavage fracture toughness testing $L[c]$ as a function of the cumulative failure probability $z$, corresponding to the scaling stress-value $\sigma_z$, and the master curve for brittle cleavage fracture toughness testing $N(c)$.

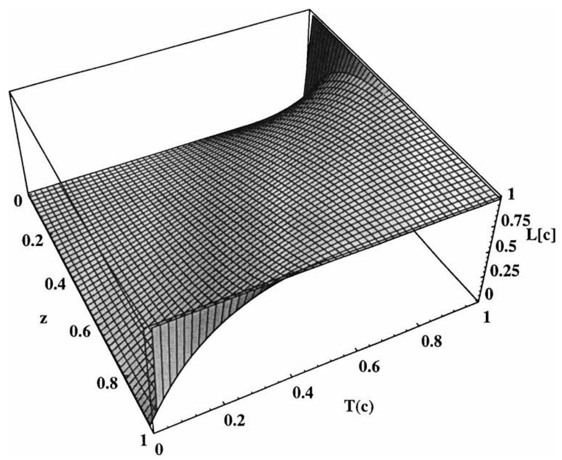


*Figure 8* The general type of master curve for brittle cleavage fracture toughness testing $L[c]$ as a function of the cumulative failure probability $z$, corresponding to the scaling stress-value $\sigma_z$, and the alternative master curve for brittle cleavage fracture toughness testing $T(c)$.

$$L[\overline{d}(z)] = 1 - (1-z)^{\{-4\,ln(1-z)\int_0^\infty [d(z)]^4 (1-z)^{[d(z)]^4}\, d[d(z)]\}^4}$$
$$= 1 - \exp\left\{-\left[\left(\Gamma\left(1+\frac{1}{4}\right)\right)^4\right]\right\} \approx 0.491 \quad (48)$$

Finally, by setting $d(z) = a = b = c =$ constant, the following relations between the master curves $L[d(z)]$, $N(a)$, and $T(b)$ are found:

$$L[c] = 1 - \exp\left\{-\frac{4}{3}\,ln(1-z)\,ln[1-N(c)]\right\} \quad (49)$$

$$L[c] = 1 - \exp\left\{\frac{[-ln(1-z)]\,ln[1-T(c)]}{\left[\Gamma\left(1+\frac{1}{4}\right)\right]^4}\right\} \quad (50)$$

Equations 49 and 50 are displayed in Figs 7 and 8.

## 5. Discussion and conclusions

The theoretical and experimental apparent fracture toughness master curves $L[d(z)]$ and $L\exp[d(z)]$ represent cumulative failure probability distribution functions, in terms of a scaled, dimensionless stress intensity $d(z)$. In the case of pure, brittle cleavage fracture, Equation 36 merely represents a mathematical link between the Weibull theory and the Wallin theory. However, if it is applied to the case of materials undergoing an amount of stable crack growth prior to failure, it is very useful because it supplys characteristic deviation magnitudes as will be shown in detail in part 2 of this series of papers. Equation 36 enables a mathematical elimination of the influence of the initial defect-size distribution. Therefore, the combined effects on the cumulative failure probability distribution, created by the stress-fields ahead of the large cracks, in interaction with the active mechanisms controlling stable growth and nucleation of (micro-)cracks, are revealed.

The presented quasi-static Weibull-Wallin theory for master curves will be extended to dynamic loading conditions as obtained by performing Charpy impact tests in part 3 of this series of papers. Therefore, instrumented Charpy impact tests have been performed with ferritic/martensitic steels. The testing-temperature has been selected close to the ductile-to-brittle transition temperature (DBTT) in order to get relevant data with respect to crack-tip shielding, stable crack growth, and capacity of microcracking prior to final rupture.

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